8. SEDOV L.I., Reflections on science and scientists (Razmyshleniya o nauke i ob uchenykh), NAUKA, Moscow, 1980.
9. BETYAEV S.K. and GAIFULLIN A.M., The large-scale structure of the nucleus of a spiral discontinuity in a liquid, Izv. Akad. Nauk SSSR, MZhG, No.5, 1982.

Translated by R.C.G.

*PMM U.S.S.R.*,Vol.48,No.1,pp.102-104,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00


# THE FIRST FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY FOR A SYMMETRIC LUNE*

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The first fundamental problem of the theory of elasticity is considered for a symmetric lune, when a symmetrically distributed normal load is specified on its boundary, and there are no tangential stresses. The problem is formulated and solved without preliminary reduction to the basic biharmonic problem. The proposed version and solution are based on the combined method of Fourier integrals and analysis of the Carleman problem /1, 2/. The problem of the stress state in a circular lune acted upon along the segments of its side surface by a uniform, normal compressive force was considered earlier in /3/, where the first fundamental problem of the theory of elasticity for a lune was reduced to the corresponding biharmonic problem.

1. The problem is formulated as follows /3/: to find the solution of the boundary value problem

$$\left[\frac{\partial^4}{\partial\alpha^4}+2\frac{\partial^4}{\partial\alpha^2\partial\beta^2}+\frac{\partial^4}{\partial\beta^4}-2\frac{\partial^2}{\partial\alpha^2}+2\frac{\partial^2}{\partial\beta^2}+1\right]\frac{\Phi}{h}=0 \tag{1.1}$$
$$-\infty<\alpha<\infty,\quad -\gamma<\beta<\gamma$$

$$\left[\frac{1}{h}\frac{\partial^2}{\partial\alpha^2}-\operatorname{sh}\alpha\frac{\partial}{\partial\alpha}+\sin\beta\frac{\partial}{\partial\beta}-\cos\beta\right]\frac{\Phi}{h}=-\frac{q(\alpha)}{2} \tag{1.2}$$
$$-\infty<\alpha<\infty,\quad \beta=\pm\gamma$$
$$\frac{\partial^2}{\partial\alpha\,\partial\beta}\left[\frac{\Phi}{h}\right]=0,\quad \beta=\pm\gamma;\quad h=\frac{1}{\operatorname{ch}\alpha+\cos\beta}$$

where $g(\alpha)/2$ is a given function characterizing the distributed load, and $\Phi(\alpha,\beta)$ is an unknown function. The symmetry of the stress state makes it possible to utilize the boundary conditions on the coordinate line $\beta=\gamma$ only, and we need consider only half of the region occupied by the lune $-\infty<\alpha<\infty,\ 0\leqslant\beta\leqslant\gamma$.

Applying the integral Fourier transformation to (1.1) and boundary conditions (1.2), we obtain

$$\frac{d^4W}{d\beta^4}+2(1-x^2)\frac{d^2W}{dx^2}+(x^2+1)^2W=0 \tag{1.3}$$

$$(x+i)^2W(x+i,\gamma)+(x-i)^2W(x-i,\gamma)+2\cos\gamma x^2W(x,\gamma)+ \tag{1.4}$$
$$i(x+i)W(x+i,\gamma)-i(x-i)W(x-i,\gamma)=G(x)$$
$$\left.\frac{dW}{d\beta}\right|_{\beta=\gamma}=0$$
$$\left(W=V\left(\frac{\Phi}{h}\right),\ G=V(2q),\ V(f)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(\alpha)e^{i\alpha x}d\alpha\right)$$

Let us write the general solution of (1.3) symmetrical with respect to $\beta$

$$W(x,\beta)=A(x)\operatorname{ch}x\beta\cos\beta+B(x)\operatorname{sh}x\beta\sin\beta \tag{1.5}$$

Substituting this solution into the second boundary condition of (1.4), we obtain a relation connecting $B(x)$ with $A(x)$, and from (1.5) we obtain

$$W(x,\beta)=A(x)[\operatorname{ch}x\beta\cos\beta+C(x)\operatorname{sh}x\beta\sin\beta] \tag{1.6}$$
$$C(x)=(\operatorname{tg}\gamma-x\operatorname{th}x\gamma)(\operatorname{tg}\gamma-\operatorname{th}x\gamma)^{-1}$$

*Prikl.Matem.Mekhan.,48,1,149-152,1984

Using the first boundary condition of (1.4), we arrive at

$$(x+2i)(x+i)F(x+i)+(x-2i)(x-i)F(x-i)+ 2\cos\gamma(x^2+1)F(x)=G(x) \quad (F(x)=W(x,\gamma)) \tag{1.7}$$

2. We shall solve (1.7) by three different methods.

*First method*. Multiplying (1.7) by $x$ we obtain

$$(x^2+2xi)\Phi_1(x+i)+(x^2-2xi)\Phi_1(x-i)+2\cos\gamma(x^2+1)\Phi_1(x)=xG(x) \quad (\Phi_1(x)=xF(x)). \tag{2.1}$$

Consider a more general equation

$$\Big(\sum_{k=0}^{n} a_k x^k\Big)\Phi_1(x+i)+\Big(\sum_{k=0}^{n} b_k x^k\Big)\Phi_1(x-i)+\lambda\Big(\sum_{k=0}^{n} c_k x^k\Big)\Phi_1(x)=xG(x) \tag{2.2}$$

whose coefficients have the following properties:

$$b_{n-r}=a_{n-2p}-\frac{2}{n-2p}\sum_{k=0}^{n}\frac{(n-2k-1)!}{(r-2k-1)!}A_{n-2k-1}i^{r-2k-1}, \quad r=2p \tag{2.3}$$

$$b_{n-r}=a_{n-2p-1}-\frac{2}{n-2p-1}\sum_{k=0}^{n}\frac{(n-2k)!}{(r-2k)!}A_{n-2k}i^{r-2k}, \quad r=2p+1$$

$$c_{n-r}=A_{n-r}, \quad r=0,1,\ldots,n$$

$$\Big(A_{n-r}=a_{n-r}-\frac{1}{(n-r)}\sum_{k=1}^{n}\frac{(n-r+k)!}{k!}A_{n-r+k}i^{k}\Big)$$

The properties of the coefficients (2.3) enable us to write (2.2) in the form

$$\Psi(x+i)+\Psi(x-i)+\lambda\Psi(x)=xG(x) \tag{2.4}$$

$$\Psi(x)=\Big[\sum_{k=0}^{n}A_k x^k\Big]\Phi_1(x)$$

Applying now an inverse Fourier transformation to (2.4), we obtain

$$\Phi_1(x)=\Big[\sum_{k=0}^{n}A_k x^k\Big]^{-1}V\Big(\frac{iq'(x)}{2\operatorname{ch}x+\lambda}\Big) \tag{2.5}$$

A solution of (2.1) is obtained from (2.5) at $\lambda=2\cos\gamma$ and $A_0=A_2=1$, $A_1=0$. Finally, we write the solution of (1.7) in the form

$$F(x)=\frac{1}{x(x^2+1)}V\Big(\frac{iq'(x)}{2\operatorname{ch}x+2\cos\gamma}\Big)$$

*Second method*. Applying to (1.7) an inverse Fourier transformation, we obtain a differential equation with variable coefficients

$$-(e^{-x}+e^{x}+2\cos\gamma)f''(x)+(e^{x}-e^{-x})f'+2\cos\gamma f(x)=q(x) \tag{2.6}$$

We will seek a solution in the form

$$f(x)=Ae^{\alpha x}+C$$

and find two linearly independent solutions of the homogeneous equation

$$f_{\pm}(x)=e^{\pm x}+\frac{1}{\cos\gamma}, \quad \gamma\neq\frac{\pi}{2}$$

If $\gamma=\pi/2$, then the general solution of the homogeneous equation (2.6) is easily obtained by reducing the order of the equation. We shall not consider this case, since when $\gamma=\pi/2$, the lune becomes a circle for which the first fundamental problem can be solved more simply. When $\gamma\neq\pi/2$, the general solution of (2.6) can be written in the form

$$f(x)=c_1f_+(x)+c_2f_-(x)+g(x) \tag{2.7}$$

where $g(x)$ is a particular solution which can be found by varying the constants. Applying a Fourier transformation to (2.7), we obtain

$$F(x)=\frac{c_1}{\sqrt{2\pi}}\delta(x-i)+\frac{c_1+c_2}{\sqrt{2\pi}}\delta(x)+\frac{c_2}{\sqrt{2\pi}}\delta(x+i)+V(g) \tag{2.8}$$

*Third method*. Applying a Fourier transformation to (2.4) we obtain a differential equation with constant coefficients

$$\sum_{k=0}^{n}i^kA_k\varphi_1^{(k)}(x)=\frac{iq'(x)}{2\operatorname{ch}x+\lambda} \tag{2.9}$$

The corresponding equation for (2.1) takes a simpler form, namely

$$-\varphi_1''(x)+\varphi_1(x)=\frac{iq'(x)}{2\operatorname{ch}x+2\cos\gamma} \tag{2.10}$$

The general solution of the homogeneous equation (2.10) is

$$\varphi_1(x)=c_1e^x+c_2e^{-x}$$

and the particular solution $g(x)$ of the inhomogeneous equation can be found by varying the constants. Applying the Fourier transformation to the general solution of (2.10), we obtain

$$F(x)=\frac{1}{x}\left[\frac{c_1}{\sqrt{2\pi}}\delta(x-i)+\frac{c_2}{\sqrt{2\pi}}\delta(x+i)+\frac{1}{x}V(g)\right] \tag{2.11}$$

We can confirm by direct substitution that the linear combinations of the delta functions appearing in (2.8) and (2.11) satisfy the homogeneous functional equation (2.7). The function $F(x)$ obtained by the second and third method yields the solution of the functional equation (1.7) in a wider class.

*Note.* The function $f(x)=V^{-1}(F)$ has a well-defined physical meaning, namely $f(x)$ describes the principal vector of the forces applied to the arcs $\beta=\pm\gamma$ /3/. If the constants are arbitrary, the components of the force vector may tend to infinity at the corner points ($\alpha=\pm\infty, \beta=\pm\gamma$) From the mathematical point of view this implies that the solution of the problem in question will not be unique. For the solution to be unique, the boundary conditions must be supplmented by additional conditions as was done in /4/.

3. Having determined the function $F(x)$, we use (1.6) to obtain $W(x,\beta)$

$$W(x,\beta)=\frac{2F(x)}{x\sin 2\gamma x+\operatorname{sh}2\gamma x}\left[(x\operatorname{ch}x\gamma+\sin\gamma+\operatorname{sh}x\gamma\cos\gamma)\operatorname{ch}x\beta\cos\beta+(\operatorname{ch}x\gamma\sin\gamma-x\operatorname{sh}x\gamma)\operatorname{sh}x\beta\sin\beta\right] \tag{3.1}$$

The function $(\Phi/h)(\alpha,\beta)$ is obtained using the formula

$$\left(\frac{\Phi}{h}\right)(\alpha,\beta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}W(x,\beta)e^{-i\alpha x}\,dx \tag{3.2}$$

The integral in (3.2) is found using the method of residues. A table (/3/, p.61) may be used to find the singularities. We must also remember that the function $F(x)$ yields its own singularities. Having found $(\Phi/h)(\alpha,\beta)$, we use the well-known formulas (/3/, p.61) to find the tangential and normal stresses. We will consider two special cases:

1) $q(x)=p\delta(x)$. Then, according to (2.4)

$$F(x)=\frac{1}{x(x^2+1)}V(D),\quad D(x)=\frac{ip\delta'(x)}{2\operatorname{ch}x+2\cos\gamma}$$

and we find

$$F(x)=-[\sqrt{2\pi}(x^2+1)(2\cos\gamma+2)]^{-1}$$

2) $q(x)=p=\text{const}$. Solving the problem by any of the above methods and taking into account the note given above, we obtain

$$F(x)=\frac{p\delta(x)}{\sqrt{2\pi}\cos\gamma}$$

and from (3.1), (3.2) we find

$$\left(\frac{\Phi}{h}\right)(\alpha,\beta)=p\frac{\cos\beta\sin\gamma+\gamma\cos\gamma\cos\beta+\beta\sin\gamma\sin\beta}{8\pi\cos\gamma(\cos\gamma\sin\gamma+\gamma)}.$$

REFERENCES


1. KARELIN A.A. and KEREKESHA P.V., On the theory of the Carleman problem for a strip, with an analytic shear displacement into the region. Dop. Akad. Nauk UkrSSR, ser.A, No.12, 1975.
2. DASHCHENKO A.F., KEREKESHA P.V. and POPOV G.YA., On a method of solving the problems of the torsion of reinforced rods. Prikl. Mekhanika, Vol.13, No.6, 1977.
3. UFLIAND IA.S., Integral Transforms in Problems of the Theory of Elasticity. Leningrad, NAUKA, 1967.
4. MUSKHELISHVILI N.I., Some Fundamental Problems of the Mathematical Theory of Elasticity. Moscow, Iz-vo Akad. Nauk SSSR, 1954.


Translated by L.K.